Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 20 (2007) 723–728
Applied Mathematics Letters
www.elsevier.com/locate/aml

Internal waves caused by the interaction of three harmonics under no restoring forces

M.C.W. Jones

School of Mathematics, Kingston University, Kingston-upon-Thames KT1 2EE, UK

Received 10 September 2004; received in revised form 12 July 2005; accepted 11 April 2006

Abstract

The physical system under consideration consists of two fluids in uniform horizontal laminar motion parallel to their interface. All external restoring forces are absent. We find that nevertheless, three-wave interaction between different harmonics of the motion can take place. Attention is focused on the situation when this resonance is among the lowest three harmonics. Using the method of strained coordinates, a system of three coupled nonlinear partial differential equations is derived. These model the propagation of the interface, correct to second order. Solutions are found whose stability is briefly considered.

Keywords: Fluid–fluid interface; Resonant waves; Evolution equations; Stability

1. Introduction

As far back as 1915, it was noticed [20] that in the theory of capillary–gravity waves, the first two harmonics of a fundamental mode can travel at the same speed and thus induce a resonant interaction. Over the intervening years this result had been generalized [3,5–7,13] to allow for other interactions between different modes of the motion. This too has been generalized [8,9] to the case when the interaction takes place on the interface between two fluids, rather than on a free surface; and even to other physical systems such as hot electron plasma [17]. This builds on the classical and celebrated result of the Kelvin–Helmholtz instability which states that, in the linear theory, the interface between two fluids is unstable provided the tangential velocities are unequal. However, certainly in the theory of water waves it is always true that both the restoring forces of capillarity and gravitation are required for resonance to be present; if either is absent then resonance cannot occur. However, in this note, we show that in the case of internal waves, if both forces are absent, so that the motion is driven solely by mean flow effects, then a triple interaction between any three harmonics of the motion is in fact possible. This is worth pointing out because while triple (and higher) resonances are not unheard of (see [1–3,12] for instance), such interactions are nevertheless relatively rare, certainly in the area of capillary–gravity waves. In fact Okamoto showed in [18] that a triple interaction cannot arise in the theory of finite depth free surface flows and later in [19] he demonstrated that even the introduction of another fluid of finite vertical extent cannot produce three interacting harmonics. A situation such as that discussed in this work

E-mail address: Mark.Jones@kingston.ac.uk.

0893-9659/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2006.04.026

could occur when two highly dense (that is relative to capillarity and gravity) almost miscible fluids (turpentine and silicon oil for example) are both moving at high speeds (i.e. the Reynolds number is large), perhaps in an astrophysical setting.

2. The evolution equations

Let us consider the situation of two stratified ideal fluids in uniform horizontal laminar irrotational motion parallel to their interface. We pick an x–y coordinate system so that the undisturbed interface occupies $y = 0$ and disturbances are represented by the (*a priori* unknown) function $y = \eta(x, t)$. The potential in the lower (upper) fluid is given by $\varphi_1(x, y, t)$, $(\varphi_2(x, y, t))$.

Classically, the equations of motion are then

$$\nabla^2\varphi_1 = 0, \quad y \le \eta, \tag{2.1a}$$

$$\nabla^2\varphi_2 = 0, \quad y \ge \eta, \tag{2.1b}$$

$$\varphi_1 \to 0, \quad y \to -\infty, \tag{2.1c}$$

$$\varphi_2 \to 0, \quad y \to \infty, \tag{2.1d}$$

$$\eta_t - \varphi_{jy} + U_j\eta_x + \varphi_{jx}\eta_x = 0, \quad y = \eta,\ j = 1, 2 \tag{2.1e}$$

$$\begin{aligned}&\rho_2\varphi_{2t} - \rho_1\varphi_{1t} + \rho_2U_2\varphi_{2x} - \rho_1U_1\varphi_{1x} + (\rho_2 - \rho_1)g\eta\\ &\quad + \frac{\rho_2}{2}(\varphi_{2x}^2 + \varphi_{2y}^2) - \frac{\rho_1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2) + S\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} = 0, \quad y = \eta.\end{aligned} \tag{2.1f}$$

In these equations, U_1 and U_2 denote the speeds in the lower and upper fluids; $\rho_1$and ρ_2 the densities; g and S the restoring forces of gravity and surface tension.

It is quite well known that for certain distinguished values of the parameters, the linearized equations can admit a solution space spanned by two distinct harmonics of the motion, but that if we consider just gravity (or capillary) waves then the solution space is always one dimensional. However in this paper we show that if both restoring forces are absent, then interactions between three harmonics of the motion are in fact possible. This means that Bernoulli's condition (2.1f) is applied in the form

$$\rho_2\varphi_{2t} - \rho_1\varphi_{1t} + \rho_2U_2\varphi_{2x} - \rho_1U_1\varphi_{1x} + \frac{\rho_2}{2}(\varphi_{2x}^2 + \varphi_{2y}^2) - \frac{\rho_1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2) = 0, \quad y = \eta. \tag{2.2}$$

Let us first linearize the boundary conditions; they become

$$\eta_t - \varphi_{jy} + U_j\eta_x = 0, \quad y = 0,\ j = 1, 2 \tag{2.3a}$$

$$\rho_2\varphi_{2t} - \rho_1\varphi_{1t} + \rho_2U_2\varphi_{2x} - \rho_1U_1\varphi_{1x} = 0, \quad y = 0. \tag{2.3b}$$

Let us seek solutions which are the Lth, Mth and Nth harmonics of a fundamental with wavenumber k and frequency ω. So set

$$\varphi_1 = A_nE(n)\exp(nky), \qquad \varphi_2 = B_nE(n)\exp(-nky), \qquad \eta = C_nE(n), \tag{2.4}$$

where $E(n) = \exp\mathrm{i}n(kx - \omega t)$, $n = L, M, N$.

It is straightforward to verify that the two kinematic conditions (2.3a) are satisfied provided that

$$A_n = \mathrm{i}(U_1 - \omega/k)C_n \quad \text{and} \quad B_n = -\mathrm{i}(U_2 - \omega/k)C_n, \quad n = L, M, N. \tag{2.5}$$

Then substituting into (2.3b) yields $\mathrm{i}n\rho_2(U_2k - \omega)B_n + \mathrm{i}n\rho_1(U_1k - \omega)A_n = 0$, which upon substituting in (2.5) yields the dispersion relation

$$\rho_2(U_2k - \omega)^2 + \rho_1(U_1k - \omega)^2 = 0. \tag{2.6}$$

Before continuing, let us digress slightly and consider what happens for nonzero surface tension. If we substitute (2.4) and (2.5) into the linearization of (2.1f) we obtain, after some algebra

$$\rho(U_2k - \omega)^2 + (U_1k - \omega)^2 = \frac{gk}{n}(1 - \rho) + \frac{S}{\rho_1}n^2k^3. \tag{2.7}$$

Now, if this is to be true for three values of n, that is $n = L, M, N$ say, then, since the left hand side is independent of n, we must have

$$\frac{Sk^2}{\rho_1} = \frac{g(1-\rho)}{MN} = \frac{g(1-\rho)}{LN} = \frac{g(1-\rho)}{LM}. \tag{2.8}$$

But if S is nonzero, then $L = M = N$. Hence we see that triple resonance cannot occur even for small values of S; any perturbation from zero destroys it. Essentially this is a singular perturbation problem.

3. Nonlinear theory

In this section we look at the nonlinear evolution of the waves produced by the interaction of the first three adjacent harmonics; that is, in the notation of the last section we shall set $(L, M, N) = (1, 2, 3)$. This reaction is of particular interest mathematically because the interactions occur at quadratic level in this situation as opposed to the more commonly encountered cubic level. Indeed a resonant interaction between any two of the waves may occur in this scenario since we see that $1 + 2 = 3$, $3 - 2 = 1$ and $3 - 1 = 2$ reactions are all possible. The situation considered here is also important physically because interactions are more likely to occur between waves whose wavenumbers are close together than ones for which they are far apart. Indeed, since the $(1, 2, 3)$ interaction is heavily quadratic, we would assume that this is the one most likely to occur in nature or be realizable experimentally.

Our main tool in this section will be the method of multiple scales, also known as the Lindstedt–Poincare method. For detailed descriptions of this technique see, for example, the books [4,16]; while other examples of its use in water wave problems may be found in [8,9,15,17]. The fundamental idea is to develop a weakly nonlinear theory by introducing a small positive parameter ε which acts as a measure of the interface steepness. We then introduce the 'slow variables' $X = \varepsilon x$ and $T = \varepsilon t$. The next step is to assume that the coefficients A_n, B_n and C_n, found in the last section, are not constants but instead are functions of these slow variables. To proceed, we expand the velocity potentials and the interface profile as power series in ε. Then making use of (2.5), they become, up to second order (where for ease of notation we write $V_i = U_i - \omega/k$, $i = 1, 2$ and c.c. denotes complex conjugate),

$$\begin{aligned}
\varphi_1 = & \left[\varepsilon \mathrm{i} V_1 C_1 + \varepsilon^2 y V_1 C_{1X} + \varepsilon^2 A_1^{(2)}\right] E(1)\mathrm{e}^{ky} \\
& + \left[\varepsilon \mathrm{i} V_1 C_2 + \varepsilon^2 y V_1 C_{2X} + \varepsilon^2 A_2^{(2)}\right] E(2)\mathrm{e}^{2ky} \\
& + \left[\varepsilon \mathrm{i} V_1 C_3 + \varepsilon^2 y V_1 C_{3X} + \varepsilon^2 A_3^{(2)}\right] E(3)\mathrm{e}^{3ky} \\
& + \varepsilon^2 A(4) E(4)\mathrm{e}^{4ky} + \varepsilon^2 A(5) E(5)\mathrm{e}^{5y} + \varepsilon^2 A(6) E(6)\mathrm{e}^{6ky} + \text{c.c.}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\varphi_2 = & \left[-\varepsilon \mathrm{i} V_2 C_1 + \varepsilon^2 y V_2 C_{1X} + \varepsilon^2 B_1^{(2)}\right] E(1)\mathrm{e}^{-ky} \\
& + \left[-\varepsilon \mathrm{i} V_2 C_2 + \varepsilon^2 y V_2 C_{2X} + \varepsilon^2 B_2^{(2)}\right] E(2)\mathrm{e}^{-2ky} \\
& + \left[-\varepsilon \mathrm{i} V_2 C_3 + \varepsilon^2 y V_2 C_{3X} + \varepsilon^2 B_3^{(2)}\right] E(3)\mathrm{e}^{-3ky} \\
& + \varepsilon^2 B(4) E(4)\mathrm{e}^{-4ky} + \varepsilon^2 B(5) E(5)\mathrm{e}^{-5ky} + \varepsilon^2 B(6) E(6)\mathrm{e}^{-6ky} + \text{c.c.}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\eta = & \left[\varepsilon C_1 \mathrm{e}^{\mathrm{i}\varepsilon\sigma t} + \varepsilon^2 C_1^{(2)}\right] E(1) + \left[\varepsilon C_2 \mathrm{e}^{2\mathrm{i}\varepsilon\sigma t} + \varepsilon^2 C_2^{(2)}\right] E(2) + \left[\varepsilon C_3 \mathrm{e}^{3\mathrm{i}\varepsilon\sigma t} + \varepsilon^2 C_3^{(2)}\right] E(3) \\
& + \varepsilon^2 C(4) E(4) + \varepsilon^2 C(5) E(5) + \varepsilon^2 C(6) E(6) + \text{c.c.}
\end{aligned} \tag{3.3}$$

The reason for the occurrence of the expressions $\varepsilon \mathrm{i} V_1 C_1 + \varepsilon^2 y V_1 C_{1X}$ etc. in the expansions of φ_1 and φ_2 is to ensure that these velocity potentials satisfy Laplace's equation to the required order.

Recall that we are concerned with the behaviour of the waves both at exact resonance and also just off resonance. To achieve this goal we have introduced into the expansion of the interface profile η the detuning parameter σ which is O(1). When $\sigma = 0$ the resonance is exact; at nonzero values the interaction is just off resonance. Recall again that the coefficients C_1, $A_1^{(2)}$, $B_1^{(2)}$ etc. are all functions of the slow variables $X = \varepsilon x$ and $T = \varepsilon t$.

It will prove easier if we apply the boundary conditions on $y = 0$. To second order, they become

$$\eta_t + (V_j + \omega/k)\eta_x - \varphi_{jy} - \eta\varphi_{jyy} + \eta_x \varphi_{jx} = 0, \quad y = 0, j = 1, 2 \tag{3.4}$$

and

$$V_1^2\varphi_{2t} + V_1^2(V_2 + \omega/k)\varphi_{2x} + V_2^2\varphi_{1t} + V_2^2(V_1 + \omega/k)\varphi_{1x}$$
$$+ V_1^2\eta[\varphi_{2ty} + (V_2 + \omega/k)\varphi_{2xy}] + V_2^2\eta[\varphi_{1ty} + (V_1 + \omega/k)\varphi_{1xy}]$$
$$+ \frac{V_1^2}{2}(\varphi_{2x}^2 + \varphi_{2y}^2) + \frac{V_2^2}{2}(\varphi_{1x}^2 + \varphi_{1y}^2) = 0, \quad y = 0, \tag{3.5}$$

where we have made use of the dispersion relation at the end of Section 2.

The next step is to substitute the expansions (3.1)–(3.3) into the conditions (3.4) and (3.5) and match successive powers of ε. For other instances of this technique see [8,9,15]. The terms of order ε are matched already. When we match terms of the form $\varepsilon^2 E(1)$ the two kinematic conditions yield

$$\mathrm{i}kV_1C_1^{(2)} + C_{1T} + (\omega/k)C_{1X} - kA_1^{(2)} + \mathrm{i}\sigma C_1 + 3\mathrm{i}k^2V_1C_2C_1^* - 5\mathrm{i}k^2V_1C_3C_2^* = 0 \tag{3.6a}$$

and

$$\mathrm{i}kV_2C_1^{(2)} + C_{1T} + (\omega/k)C_{1X} + kB_1^{(2)} + \mathrm{i}\sigma C_1 + 3\mathrm{i}k^2V_2C_2C_1^* + 5\mathrm{i}k^2V_2C_3C_2^* = 0, \tag{3.6b}$$

where the asterisk denotes complex conjugate.

Bernoulli's condition then gives us

$$\mathrm{i}kV_1B_1^{(2)} + \mathrm{i}kV_2A_1^{(2)} + \mathrm{i}(V_2 - V_1)C_{1T} + \mathrm{i}(\omega/k)(V_2 - V_1)C_{1X} - k^2V_1V_2C_2C_1^* - 2k^2V_1V_2C_3C_2^* = 0. \tag{3.6c}$$

Putting these together, we obtain

$$\mathrm{i}(V_2 - V_1)C_{1T} + \mathrm{i}(\omega/k)(V_2 - V_1)C_{1X} + 2k^2V_1V_2C_2C_1^* + 4k^2V_1V_2C_3C_2^* + (\sigma/2)(V_2 - V_1)C_{1T} = 0. \tag{3.7}$$

Clearly two analogous equations can be derived by consideration of the terms of order $\varepsilon^2 E(2), \varepsilon^{2\cdot} E(3)$. The dimensional quantities V_1, V_2, ω and k may be eliminated and the equations thus simplified by means of a succession of scalings (for other examples of this, see [8,10]). First we write

$$\sigma(V_1 - V_2) \to \sigma, \qquad X \to (\omega/k)(V_2 - V_1)X, \qquad T \to (V_2 - V_1)T \quad \text{and} \quad \sigma \to k^2V_1V_2\sigma.$$

We continue by setting $X \to X/(k^2V_1V_2)$ and $T \to T/(k^2V_1V_2)$.

The upshot is then the three equations

$$\mathrm{i}C_{1T} + \mathrm{i}C_{1X} + 2C_2C_1^* + 4C_3C_2^* + (\sigma/2)C_1 = 0, \tag{3.8a}$$
$$\mathrm{i}C_{2T} + \mathrm{i}C_{2X} + C_1^2 + 4C_3C_1^* + \sigma C_2 = 0, \tag{3.8b}$$
$$\mathrm{i}C_{3T} + \mathrm{i}C_{3X} + 4C_1C_2 + (3\sigma/2)C_3 = 0. \tag{3.8c}$$

The three coupled nonlinear equations (3.8) are the evolution equations for an interface formed by the interaction of the first three harmonics of a fundamental. They may be compared with the systems found in [12–14] which describe the propagation of an interface formed by the interaction between the first two harmonics of the motion. That system was also of quadratic order but in that case the forces of surface tension and gravity were also taken into account.

4. Solving the equations

We seek Stokes-type solutions in the form

$$C_1 = \exp\{\mathrm{i}lX + \mathrm{i}\gamma T\}, \tag{4.1a}$$
$$C_2 = g_2\exp\{2(\mathrm{i}lX + \mathrm{i}\gamma T)\}, \tag{4.1b}$$
$$C_3 = g_3\exp\{3(\mathrm{i}lX + \mathrm{i}\gamma T)\}. \tag{4.1c}$$

Substituting into the system leads us to

$$4g_2 + 8g_2g_3 - K = 0, \tag{4.2a}$$
$$1 + 4g_3 - Kg_2 = 0, \tag{4.2b}$$
$$8g_2 - 3Kg_3 = 0, \tag{4.2c}$$

where $K = 2\gamma + 2l - \sigma$. Rearranging the third of these equations yields $K = 8g_2/3g_3$ which when substituted into the first yields a quadratic in g_3. Having solved for g_3 we may then find g_2 from the middle equation and thence K. The solutions turn out to be

$$(g_3, g_2, K) = \left(\frac{-3+\sqrt{57}}{12}, \pm\frac{\sqrt{19-\sqrt{57}}}{4\sqrt{2}}, \pm\frac{(3+\sqrt{57})(\sqrt{19-\sqrt{57}})}{6\sqrt{2}} \right) \tag{4.3a}$$
$$(g_3, g_2, K) = \left(\frac{-3-\sqrt{57}}{12}, \pm\frac{\sqrt{19+\sqrt{57}}}{4\sqrt{2}}, \pm\frac{(3-\sqrt{57})(\sqrt{19+\sqrt{57}})}{6\sqrt{2}} \right). \tag{4.3b}$$

Notice that there are four sets of solutions (of course $K = g_2 = 0$, $g_3 = -1/4$ is also a solution, but not relevant for our purposes). We may proceed to consider the plane wave stability of these solutions. Impose perturbations and write

$$C_1 = (1 + p_1)\exp\{\mathrm{i}lX + \mathrm{i}\gamma T + \mathrm{i}\theta_1\} \tag{4.4a}$$
$$C_2 = g_2(1 + p_2)\exp\{2(\mathrm{i}lX + \mathrm{i}\gamma T + \mathrm{i}\theta_2)\} \tag{4.4b}$$
$$C_3 = g_3(1 + p_3)\exp\{3(\mathrm{i}lX + \mathrm{i}\gamma T + \mathrm{i}\theta_3)\}. \tag{4.4c}$$

Then substituting into (3.8) and ignoring products of small quantities leads to the following equations:

$$4g_2g_3p_1 - (2g_2 + 4g_2g_3)p_2 - 4g_2g_3p_3 + \theta_{1T} + \theta_{1X} = 0 \tag{4.5a}$$
$$p_{1T} + p_{1X} - (4g_2 + 4g_2g_3)\theta_1 + (2g_2 - 4g_2g_3)\theta_2 + 4g_2g_3\theta_3 = 0 \tag{4.5b}$$
$$(4g_3 + 2)p_1 - (4g_3 + 1)p_2 + 4g_3p_3 - g_2(\theta_{2T} + \theta_{2X}) = 0 \tag{4.5c}$$
$$g_2(p_{2T} + p_{2X}) + (2 - 4g_3)\theta_1 - (4g_3 + 1)\theta_2 + 4g_3\theta_3 = 0 \tag{4.5d}$$
$$4g_2p_1 + 4g_2p_2 - 4g_2p_3 - g_3(\theta_{3T} + \theta_{3X}) = 0 \tag{4.5e}$$
$$g_3(p_{3T} + p_{3X}) + 4g_2\theta_1 + 4g_2\theta_2 - 4g_2\theta_3 = 0. \tag{4.5f}$$

We shall assume plane wave perturbations, so that

$$\begin{pmatrix} p_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \overline{p_i} \\ \overline{\theta}_i \end{pmatrix} \exp\{\mathrm{i}\delta X + \mathrm{i}\kappa T\}. \tag{4.6}$$

Then substituting into (4.5) we obtain a set of equations for the perturbation amplitudes which can only be consistent if the following determinant vanishes:

$$\begin{vmatrix} -\mathrm{i}(\kappa+\delta) & -4g_2g_3 & 0 & 2g_2+4g_2g_3 & 0 & 4g_2g_3 \\ -4(g_2+g_2g_3) & \mathrm{i}(\kappa+\delta) & (2g_2-4g_2g_3) & 0 & 4g_2g_3 & 0 \\ 0 & 4g_3+2 & -\mathrm{i}g_2(\kappa+\delta) & -(1+4g_3) & 0 & 4g_3 \\ 2-4g_3 & 0 & -(4g_3+1) & \mathrm{i}g_2(\kappa+\delta) & 4g_3 & 0 \\ 0 & 4g_2 & 0 & 4g_2 & -\mathrm{i}g_3(\kappa+\delta) & -4g_2 \\ 4g_2 & 0 & 4g_2 & 0 & -4g_2 & \mathrm{i}g_3(\kappa+\delta) \end{vmatrix}. \tag{4.7}$$

When $g_3 = (-3 + \sqrt{57})/12$ and $g_2 = \pm\sqrt{19 - \sqrt{57}}/4\sqrt{2}$, expanding the determinant gives us

$$180\sqrt{57}\kappa^6 - 1596\kappa^6 - 11\,077\sqrt{57}\kappa^4 + 109\,079\kappa^4 - 363\,888\sqrt{57}\kappa^2 + 2339\,280\kappa^2 = 0, \tag{4.8}$$

which, apart from zero, has solutions ± 4.429 and ± 9.37. Hence the interfaces are *stable*. When $g_3 = (-3-\sqrt{57})/12$ and $g_2 = \pm\sqrt{19+\sqrt{57}}/4\sqrt{2}$ the equation corresponding to (4.8) has exactly the same rational coefficients, but the irrational ones have opposite signs. The nonzero solutions are $\pm 4.429\mathrm{i}$ and ± 9.37. These interfaces are hence *unstable*.

5. Conclusions

A study was made of the resonances which are possible on the interface of two ideal fluids when external restoring forces are absent, but the motion is driven by the mean flows. It was discovered that in principle, interaction between any three harmonics of a fundamental mode is possible. This contrasts with the situation when gravity and surface tension both play a part in driving the evolution of the interface, in which case only resonance between two harmonics of the motion is possible. In the scenario when only one of the forces are present, resonance cannot occur at all. A particular study was made of the interaction between the lowest three modes of the motion. In this case the coupling between the leading order amplitudes occurs at quadratic, rather than at any higher level, and it proved quite easy to derive a system of three nonlinear coupled equations which described the propagation of the interface. This system was shown to possess a number of reasonably simple Stokes-type solutions. Some were found to be stable, others not. These results may be compared with the classical theory of Rayleigh–Taylor or Kelvin–Helmholtz instabilities; for a comprehensive reference see [11]. It is shown there that in the case of nonresonant gravity waves these are catastrophic instabilities of the short wave type. Various ways in which they may be stabilized are discussed; such as the introduction of surface tension and by taking account of the effects of viscosity. Another interesting further avenue of investigation would be to find other solution types for the equations presented here, such as solitons or rational-cum-oscillatory waves.

References

[1] A.D.D. Craik, Wave Interactions and Fluid Flows, Cambridge University Press, Cambridge, 1985.
[2] A.D.D. Craik, Exact solutions of non-conservative equations for three-wave and second harmonic resonance, Proc. R. Soc. Lond. A 406 (1986) 1–12.
[3] J.L. Hammack, D.M. Henderson, Resonant interactions among surface water waves, Annu. Rev. Fluid Mech. 25 (1993) 55–97.
[4] P.G. Drazin, Nonlinear-Systems, UP, Cambridge, 1994.
[5] D.M. Henderson, J.L. Hammack, Experiments on ripple instabilities. Part 1. Resonant triads, J. Fluid Mech. 184 (1987) 15–41.
[6] M.C.W. Jones, On the stability of a third harmonic resonant wavetrain, Stab. Appl. Anal. Contin. Media 2 (1992) 323–338.
[7] M.C.W. Jones, Further results on the stability of symmetric and asymmetric resonant capillary–gravity waves, Japan J. Indust. Appl. Math. 11 (1994) 465–483.
[8] M.C.W. Jones, Non-linear ripples of Kelvin–Helmholtz type which arise from an interfacial mode interaction, J. Fluid Mech. 341 (1997) 295–315.
[9] M.C.W. Jones, Nonlinear nearly Wilton ripples between two uniform stream flows, Quart. J. Mech. Appl. Math. 52 (1999) 237–252.
[10] M.C.W. Jones, A nonlinear multiple scales approach to modelling almost third harmonic interfaces, Z. Angew. Math. Phys. 50 (1999) 47–63.
[11] D.D. Joseph, J.C. Saut, Short-wave instabilities and ill-posed initial-value problems, Theor. Comput. Fluid Dyn. 50 (1990) 47–63.
[12] D.J. Kaup, Simple harmonic generation: An exact method of solution, Stud. Appl. Math. 59 (1978) 25–35.
[13] L.F. McGoldrick, On Wilton's ripples: A special case of resonant interactions, J. Fluid Mech. 42 (1970) 193–200.
[14] L.F. McGoldrick, On the rippling of small waves: A harmonic nearly resonant interaction, J. Fluid Mech. 52 (1972) 725–751.
[15] A.H. Nayfeh, Third harmonic resonance in the interaction of capillary and gravity waves, J. Fluid Mech. 48 (1971) 384–395.
[16] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.
[17] A.H. Nayfeh, S.D. Hassan, The method of multiple scales and nonlinear dispersive waves, J. Fluid Mech. 48 (1971) 463–475.
[18] H. Okamoto, On the problem of water waves of permanent configuration, Nonlinear Anal. 14 (1990) 469–481.
[19] H. Okamoto, Interfacial progressive water waves — a singularity theoretic view, Tohoku Math. J. 49 (1997) 33–57.
[20] J.R. Wilton, On ripples, Philos. Mag. 29 (1915) 688–700.